# Schrodinger Wave Equation:—

In 1926 an Austrian Physicist Erwin Schrodinger formulated a wave equation to describe the behaviour of electron wave in atom and molecules.

Consider a simple wave motion as that of the vibration of a stretched string. If $\omega$ be the amplitudes at any point whose co-ordinate is $x$ at time $t$. The appropriate form of the wave equation may be written as follows.

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{u^2} \cdot \frac{\partial^2 \omega}{\partial t^2} \quad \text{——— (i)}$$

Where '$u$' is the velocity of the propagation of the wave.
On separating the variables, this differential equation may be written as follows.

$$\omega = f(x) \cdot g(t) \quad \text{——— (ii)}$$

Where $f(x)$ is a function of the co-ordinate '$x$' only and $g(t)$ is a function of time $t$ only.
For the motion of standing waves such as occuring in a stretched string, it is possible to express $g(t)$ as,

$$g(t) = A \sin 2\pi \nu t \quad \text{——— (iii)}$$

Where $\nu$ is the vibrational frequency and $A$ is a constant, it stands for maximum amplitude and eqn (iii) gets the form (on substituting the value of $g(t)$ in eqn (ii))

$$\omega = f(x) \cdot A \sin 2\pi \nu t \quad \text{——— (iv)}$$

On differentiating the above equation with respect to '$t$' we have following eqn.

$$\frac{\partial \omega}{\partial t} = f(x) \cdot A \cos 2\pi \nu t \cdot 2\pi \nu$$

$$\frac{\partial^2 \omega}{\partial t^2} = f(x) \cdot (-) A \sin 2\pi \nu t \cdot 2\pi \nu \cdot 2\pi \nu$$

$$\frac{\partial^2 \omega}{\partial t^2} = f(x) \cdot (-) 4\pi^2 \nu^2 A \sin 2\pi \nu t \quad [\because g(t) = A \sin 2\pi \nu t]$$

$$\frac{\partial^2 \omega}{\partial t^2} = -4\pi^2 \nu^2 \cdot f(x) \cdot g(t) \quad \text{——— (v)}$$

From eqn (ii) we have, $\omega = f(x) \cdot g(t)$

$$\text{or} \quad \frac{\partial^2 \omega}{\partial x^2} = \frac{\partial^2 \{f(x)\}}{\partial x^2} \cdot g(t) \quad \text{——— (vi)}$$

Now with the help of eqn (v) and (vi) and (i), we have,

$$\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{u^2} \cdot \frac{\partial^2 \omega}{\partial t^2}$$

$$\frac{\partial^2 \{f(x)\}}{\partial x^2} \cdot \cancel{g(t)} = \frac{1}{u^2} \cdot (-4\pi^2 \nu^2) f(x) \cdot \cancel{g(t)}$$

$$\text{or} \quad \frac{\partial^2 \{f(x)\}}{\partial x^2} = -\frac{4\pi^2 \nu^2}{u^2} f(x) \quad \text{——— (vii)}$$

We know, $u = \nu \lambda$, on substituting this value in eqn (vii) we have.

$$\frac{\partial^2 \{f(x)\}}{\partial x^2} = -\frac{4\pi^2 \cancel{\nu^2}}{\cancel{\nu^2} \cdot \lambda^2} \cdot f(x)$$

$$or, \quad \frac{\partial^2 f(x)}{\partial x^2} = -\frac{4\pi^2}{\lambda^2} \cdot f(x) \quad \text{———} (viii)$$

This is the expression for the wave equation in one direction and it can be extended in three direction, expressed by the co-ordinates $x$, $y$ and $z$. $f(x)$ for one co-ordinate is replaced by $\psi(x,y,z)$ which is the amplitude function for three co-ordinates. Hence eq$^n$ (viii) may be written as

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = -\frac{4\pi^2}{\lambda^2}\psi \quad \text{———} (ix)$$

Using the symbol $\nabla^2$ (del squared) for differential operator (Laplacian-operator)

$$\text{i.e.} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

eq$^n$ (ix) may be reduced as

$$\nabla^2 \psi = -\frac{4\pi^2}{\lambda^2}\psi \quad \text{———} (x)$$

On following de-Broglie's ideas, Schrodinger, applied the above treatment to material waves associated with all particles including electron, atoms & molecules.

de-Broglie's expression, $\lambda = \frac{h}{p} = \frac{h}{mu}$

We get

$$\nabla^2 \psi = -\frac{4\pi^2}{\lambda^2}\psi$$

$$\text{or} \quad \nabla^2 \psi = -\frac{4\pi^2}{\left(\frac{h}{mu}\right)^2}\psi = -\frac{4\pi^2}{h^2} m^2 u^2 \psi \quad \text{———} (xi)$$

Where, $m$ is mass, $u$ is velocity, and $h$ is plank constant.

But we know that total energy of a particle is given by

$$E = K.E. + P.E$$

$$= \frac{1}{2}mu^2 + U$$

$$2(E-U) = mu^2 \quad \text{———} (xii)$$

On substituting this value in eq$^n$ (xi) we have.

$$\nabla^2 \psi = -\frac{4\pi^2 m}{h^2}\{2(E-U)\} \cdot \psi$$

$$\nabla^2 \psi = -\frac{8\pi^2 m}{h^2}(E-U)\psi$$

$$\text{or} \quad \nabla^2 \psi + \frac{8\pi^2 m}{h^2}(E-U)\psi = 0 \quad \text{———} (xiii)$$

The above equation is the required eq$^n$

i.e. Schrodinger's Wave Equation.